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Maxwell equations in matrix form, squaring procedure, separating the variables, and structure of electromagnetic solutions

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The Riemann – Silberstein – Majorana – Oppenheimer approach to the Maxwell electrodynamics in vacuum is investigated within the matrix formalism. The matrix form of electrodynamics includes three real 4×4 matrices: $(-i\partial_0 + \alpha^j \partial_j)\Psi(x) = 0$, where $\Psi(x) = (0, \mathbf{E}(x) + ic\mathbf{B}(x))$. Within the squaring procedure we construct four formal solutions of the Maxwell equations on the base of scalar Klein – Fock – Gordon solutions: $\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} = (i\partial_0 + \alpha^1\partial_1 + \alpha^2\partial_2 + \alpha^3\partial_3)\,\Phi(x)$, where $\Phi(x)$ satisfies equation $\partial^a\partial_a\Phi(x) = 0$. The problem of separating physical electromagnetic waves in the linear space $\{\lambda_0\Psi^0 + \lambda_1\Psi^1 + \lambda_2\Psi^2 + \lambda_3\Psi^3\}$ is investigated, several particular cases, plane waves and cylindrical waves, are considered in detail.

1 Introduction

Special relativity arose from study of the Maxwell equations symmetry with respect to motion of references frames: Lorentz [1], Poincaré [2], Einstein [3] Naturally, an analysis of the Maxwell equations with respect to Lorentz transformations was the first objects of relativity theory: Minkowski [4], Silberstein [5, 6], Marcolongo [8], Bateman [9], and Lanczos [10], Gordon [11], Mandel'stam – Tamm [12, 13, 14].

After Dirac [15] discovery of the relativistic equation for a particle with spin 1/2 much work was done to study spinor and vectors within the Lorentz group theory: Möglich [16], Ivanenko – Landau [17], Neumann [18], van der Waerden [19], Juvet [20]. As was shown any quantity which transforms linearly under Lorentz transformations is a spinor. For that reason spinor quantities are considered as fundamental in quantum field theory and basic equations for such quantities should be written in a spinor form. A spinor formulation of Maxwell equations was studied by Laporte and Uhlenbeck [21], also see Rumer [28]. In 1931, Majorana [23] and Oppenheimer [22] proposed to consider the Maxwell theory of electromagnetism as the wave mechanics of the photon. They introduced a complex 3-vector wave function satisfying the massless Dirac-like equations. Before Majorana and Oppenheimer, the most crucial steps were made by Silberstein [5], he showed the possibility to have formulated Maxwell equation in term of complex 3-vector entities. Silberstein in his second paper [6] writes that the complex form of Maxwell equations has been known before; he refers there to the second volume of the lecture notes on the differential equations of mathematical physics by B. Riemann that were edited and published by H. Weber in 1901 [7]. This not widely used fact is noted by Bialynicki-Birula [103]).

Maxwell equations in the matrix Dirac-like form considered during long time by many authors: Luis de Broglie [24, 25, 30, 36], Petiau [26], Proca [27, 44], Duffin [29], Kemmer [31, 41, 61], Bhabha [32], Belinfante [33, 34], Taub [35], Sakata – Taketani [37], Schrödinger [39, 40], Heitler [42], [45, 46], Mercier [47], Imaeda [48], Fujiwara [50], Ohmura [51], Borgardt [52, 59], Fedorov [53], Kuohsien [54], Bludman [55], Good [56], Moses [57, 60, 76], Lomont [58], Bogush – Fedorov [64], Sachs – Schwebel [66], Ellis [68], Oliver [70], Beckers – Pirotte [71], Casanova [72], Carmeli [73], Bogush [74], Lord [75], Weingarten [77], Mignani – Recami – Baldo [78], Newman [79], [80], [82], Edmonds [83], Silveira [86]; the interest to the Majorana – Oppenheimer formulation of electrodynamics has grown in recent years: Jena – Naik – Pradhan [87], Venuri [88], Chow [89], Fushchich – Nikitin [90], Cook [92, 93], Giannetto [96], Yépez, Brito –

Vargas [97], Kidd – Ardini – Anton [98], Recami [99], Krivsky – Simulik [101], Inagaki [102], Bialynicki-Birula [103, 104, 128], Sipe [105], [106], Esposito [108], Dvoeglazov [109] (see a big list of relevant references therein)-[111], Gersten [107], Kanatchikov [110], Gsponer [112], Ivezic [113, 114, 115, 116, 117, 118, 119, 120, 121], Donev – Tashkova. [125, 126, 127].

Our treatment will be with a quite definite accent: the main attention is given to possibilities given by the matrix approach for explicit constructing electromagnetic solutions of the Maxwell equations. In vacuum case, the matrix form includes three real 4×4 matrices α^j:

$$(-i\partial_0 + \alpha^j \partial_j)\Psi(x) = 0 \; ,$$

where $\Psi(x) = (0, \mathbf{E}(x) + ic\mathbf{B}(x))$. With the use of squaring procedure one may construct four formal solutions of the Maxwell equations on the base of scalar solution of the Klein – Fock – Gordon equation:

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} = (i\partial_0 + \alpha^1\partial_1 + \alpha^2\partial_2 + \alpha^3\partial_3)\; \Phi(x) \; , \qquad \partial^a \partial_a \Phi(x) = 0 \; .$$

The problem of separating physical electromagnetic solutions in the linear space $\{\lambda_0\Psi^0 + \lambda_1\Psi^1 + \lambda_2\Psi^2 + \lambda_3\Psi^3\}$ is investigated. Several particular cases are considered in detail.

2 Complex matrix form of Maxwell theory in vacuum

Let us start with Maxwell equations in vacuum [38, 63, 82, 129] (with the use of usual notation for current 4-vector $j^a = (\rho, \mathbf{J}/c)$, $c^2 = 1/\epsilon_0\mu_0$):

$$\begin{array}{ll} \text{div } c\mathbf{B} = 0 \; , & \text{rot } \mathbf{E} = -\dfrac{\partial c\mathbf{B}}{\partial ct} \; , \\ \text{div } \mathbf{E} = \dfrac{\rho}{\epsilon_0}, & \text{rot } c\mathbf{B} = \dfrac{\mathbf{j}}{\epsilon_0} + \dfrac{\partial \mathbf{E}}{\partial ct} \; , \end{array} \tag{1}$$

Let us introduce 3-dimensional complex vector $\psi^k = E^k + icB^k$, with the help of which the above equations can be combined into (see Silberschtein [5, 6], Bateman [9], Majorana [23], Oppenheimer [22], and many others)

$$\begin{array}{c} \partial_1\Psi^1 + \partial_2\Psi^0 + \partial_3\Psi^3 = j^0/\epsilon_0 \; , \;\; -i\partial_0\psi^1 + (\partial_2\psi^3 - \partial_3\psi^2) = i\, j^1/\epsilon_0 \; , \\ -i\partial_0\psi^2 + (\partial_3\psi^1 - \partial_1\psi^3) = i\, j^2/\epsilon_0 \; , \;\; -i\partial_0\psi^3 + (\partial_1\psi^2 - \partial_2\psi^1) = i\, j^3/\epsilon_0 \; . \end{array} \tag{2}$$

let $x_0 = ct$, $\partial_0 = c\partial_t$. These four relations can be rewritten in a matrix form using a 4-dimensional column ψ with one additional zero-element [90, 130]:

$$(-i\partial_0 + \alpha^j\partial_j)\Psi = J \; , \qquad \Psi = \left| \begin{array}{c} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{array} \right| , \qquad J = \frac{1}{\epsilon_0} \left| \begin{array}{c} j^0 \\ i\, j^1 \\ i\, j^2 \\ i\, j^3 \end{array} \right| ,$$

$$\alpha^1 = \left| \begin{array}{cccc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right| , \qquad \alpha^2 = \left| \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right| , \qquad \alpha^3 = \left| \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right|$$

$$(\alpha^1)^2 = -I, \qquad (\alpha^2)^2 = -I, \qquad (\alpha^2)^2 = -I,$$

$$\alpha^1\alpha^2 = -\alpha^2\alpha^1 = \alpha^3 \; , \qquad \alpha^2\alpha^3 = -\alpha^3\alpha^2 = \alpha^1 \; , \qquad \alpha^3\alpha^1 = -\alpha^1\alpha^3 = \alpha^2 \; . \tag{3}$$

3 Method to construct electromagnetic solutions from scalar ones

The above matrix form of Maxwell theory :

$$(-i\partial_0 + \alpha^j \partial_j)\Psi = 0 \, , \qquad \Psi = \left| \begin{array}{c} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{array} \right| \, . \tag{4}$$

permits us to develop a simple method of finding solutions of Maxwell equations on the base of known solutions of the scalar massless equation by Klein – Fock – Gordon. Indeed, in virtue of the above commutative relations we have an operator identity

$$(-i\partial_0 + \alpha^1\partial_1 + \alpha^2\partial_2 + \alpha^3\partial_3)\,(-i\partial_0 - \alpha^1\partial_1 - \alpha^2\partial_2 - \alpha^3\partial_3) \; = (-\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^3) \, .$$

Therefore, taking any special scalar solution $(-\partial_0^2+\partial_1^2+\partial_2^2+\partial_3^2)\;\Phi(x) = 0$, one can immediately construct four solutions of the Maxwell equation:

$$(-i\partial_0 + \alpha^1\partial_1 + \alpha^2\partial_2 + \alpha^3\partial_3)\; \Psi^a = 0 \, ,$$

where Ψ^a are columns of the matrix

$$(i\partial_0 + \alpha^1\partial_1 + \alpha^2\partial_2 + \alpha^3\partial_3)\;\Phi(x) =$$
$$= \left| \begin{array}{cccc} i\partial_0\Phi & \partial_1\Phi & \partial_2\Phi & \partial_3\Phi \\ -\partial_1\Phi & i\partial_0\Phi & -\partial_3\Phi & \partial_2\Phi \\ -\partial_2\Phi & \partial_3\Phi & i\partial_0\Phi & -\partial_1\Phi \\ -\partial_3\Phi & -\partial_2\Phi & \partial_1\Phi & i\partial_0\Phi \end{array} \right| = \{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} \, . \tag{5}$$

Thus, we have four formal solutions of the free Maxwell equations (let $F_a(x) = \partial_a\Phi(x)$):

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} = \left| \begin{array}{cccc} iF_0 & F_1 & F_2 & F_3 \\ -F_1 & iF_0 & -F_3 & F_2 \\ -F_2 & F_3 & iF_0 & -F_1 \\ -F_3 & -F_2 & F_1 & iF_0 \end{array} \right| \, . \tag{6}$$

4 Electromagnetic plane waves from scalar ones

Let us specify this method for an elementary example, starting from a scalar plane wave propagating along the axis z:

$$\Phi = A\sin(\omega t - kz) = A\sin(k_0x_0 - k_3x_3) = A\cos\varphi \, . \tag{7}$$

The recipe (5) gives

$$\{\Psi^a\} = A \left| \begin{array}{cccc} ik_0 & 0 & 0 & -k_3 \\ 0 & ik_0 & k_3 & 0 \\ 0 & -k_3 & ik_0 & 0 \\ k_3 & 0 & 0 & ik_0 \end{array} \right| \cos\varphi \, . \tag{8}$$

Because the left and the right columns have non-vanishing zero-component, they cannot represent any real solutions of the Maxwell equations. However, two remaining seem to be suitable ones (the factor $A\cos\varphi$ is omitted):

$$\Psi^{I} = \left| \begin{array}{r} 0 \\ ik_0 \\ -k_3 \\ 0 \end{array} \right| = \left| \begin{array}{c} 0 \\ E^1 + icB^1 \\ E^2 + icB^2 \\ E^3 + ciB^3 \end{array} \right| , \qquad \Psi^{II} = A \left| \begin{array}{r} 0 \\ k_3 \\ ik_0 \\ 0 \end{array} \right| = \left| \begin{array}{c} 0 \\ E^1 + icB^1 \\ E^2 + icB^2 \\ E^3 + ciB^3 \end{array} \right| .$$

Thus, we have two wave solutions:

$$I \qquad E^2 = -k_3 A \ \cos\varphi \ , \qquad B^1 = \frac{k_0}{c} A \ \cos\varphi \ ,$$

$$II \qquad E^1 = k_3 A \ \cos\varphi \ , \qquad B^2 = \frac{k_0}{c} A \ \cos\varphi \ . \tag{9}$$

In both cases, the ratio E/B equals to the speed of light:

$$\frac{E}{B} = \frac{k_3}{k_0/c} = c \ .$$

Besides, both waves have are clockwise polarized:

$$I \qquad (\mathbf{E} \times \mathbf{B}) = +\mathbf{e}_3 \ \frac{k_3 k_0}{c} A^2 \cos^2\varphi \ ,$$

$$II \qquad (\mathbf{E} \times \mathbf{B}) = +\mathbf{e}_3 \ \frac{k_3 k_0}{c} A^2 \cos^2\varphi \ . \tag{10}$$

Two waves are linearly independent and orthogonal to each other:

$$\mathbf{E}^I \mathbf{E}^{II} = 0 \ , \qquad \mathbf{B}^I \mathbf{B}^{II} = 0 \ .$$

In the same manner we can solve a more general problem of constructing plane wave solutions with arbitrary wave vector $\mathbf{k}$. Let us start with a scalar wave

$$\Phi = A \ \sin(k_0 x_0 - k_i x_i) = A \ \cos\varphi \ . \tag{11}$$

The matrix of solutions (5) will take the form

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} = A \left| \begin{array}{rrrr} ik_0 & -k_1 & -k_2 & -k_3 \\ k_1 & ik_0 & +k_3 & -k_2 \\ k_2 & -k_3 & ik_0 & +k_1 \\ k_3 & +k_2 & -k_1 & ik_0 \end{array} \right| \cos\varphi \ . \tag{12}$$

We have four formal solutions Ψ^a, but they cannot be regarded as physical because each of them has a non-vanishing zero-component. However, we can use linearity of the Maxwell equation and combine elementary columns in (12) with any coefficients. In this way we are able to construct physical solutions. For shortness let us omit the factor $A \ \sin\varphi$ and operate only with the columns of the matrix

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} = \left| \begin{array}{rrrr} ik_0 & -k_1 & -k_2 & -k_3 \\ k_1 & ik_0 & +k_3 & -k_2 \\ k_2 & -k_3 & ik_0 & +k_1 \\ k_3 & +k_2 & -k_1 & ik_0 \end{array} \right| .$$

Taking into account properties of the wave vector

$$k_0^2 - \mathbf{k}^2 = 0 \qquad \Longrightarrow \qquad \mathbf{k} = k_0 \mathbf{n} \ , \ \mathbf{n}^2 = 1 \ , \tag{13}$$

previous matrix can be rewritten as follows (the common factor k_0 is omitted)

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} = k_0 \left| \begin{array}{cccc} i & -n_1 & -n_2 & -n_3 \\ n_1 & i & +n_3 & -n_2 \\ n_2 & -n_3 & i & +n_1 \\ n_3 & +n_2 & -n_1 & i \end{array} \right| \sim \left| \begin{array}{cccc} i & -n_1 & -n_2 & -n_3 \\ n_1 & i & +n_3 & -n_2 \\ n_2 & -n_3 & i & +n_1 \\ n_3 & +n_2 & -n_1 & i \end{array} \right| .$$

First, with the help of the column (0) let us produce a zero at the first component of the columns (1) – (2) – (3):

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} \sim \left| \begin{array}{llll} i & 0 & 0 & 0 \\ n_1 & -in_1n_1 + i & -in_2n_1 + n_3 & -in_3n_1 - n_2 \\ n_2 & -in_1n_2 - n_3 & -in_2n_2 + i & -in_3n_2 + n_1 \\ n_3 & -in_1n_3 + n_2 & -in_2n_3 - n_1 & -in_3n_3 + i \end{array} \right| .$$

Noting that the column (3) is a linear combination of the columns (1) and (2):

$$i\ n_2\ (1) - i\ n_1\ (2) = (3)\ ;$$

in other words, solution (3) is a linear combination of (1) and (2). Therefore from the above it follows (multiplying the columns (1) and (2) by imaginary i):

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} \sim \left| \begin{array}{llll} i & 0 & 0 & 0 \\ n_1 & n_1^2 - 1 & n_2n_1 + in_3 & 0 \\ n_2 & n_1n_2 - in_3 & n_2n_2 - 1 & 0 \\ n_3 & n_1n_3 + in_2 & n_2n_3 - in_1 & 0 \end{array} \right| . \tag{14}$$

Thus, two physical Maxwell solutions are

$$\Psi^I = \left| \begin{array}{c} 0 \\ n_1^2 - 1 \\ n_1n_2 - in_3 \\ n_1n_3 + in_2 \end{array} \right| = \left| \begin{array}{c} 0 \\ E^1 + icB^1 \\ E^2 + icB^2 \\ E^3 + ciB^3 \end{array} \right| , \qquad \Psi^{II} = \left| \begin{array}{c} 0 \\ n_2n_1 + in_3 \\ n_2n_2 - 1 \\ n_2n_3 - in_1 \end{array} \right| = \left| \begin{array}{c} 0 \\ E^1 + icB^1 \\ E^2 + icB^2 \\ E^3 + ciB^3 \end{array} \right| , \tag{15}$$

or differently (the factor $A\ \cos\varphi$ is omitted)

$$\begin{array}{lll} I & \mathbf{E} = (n_1^2 - 1,\ n_1n_2,\ n_1n_3)\ , & c\mathbf{B} = (0,\ -n_3,\ n_2)\ ; \\ II & \mathbf{E} = (n_2n_1,\ n_2n_2 - 1,\ n_2n_3)\ , & c\mathbf{B} = (n_3,\ 0,\ -n_1)\ . \end{array} \tag{16}$$

It is the matter of simple calculation to verify the identity for amplitudes: $cB = E$. Also, both these waves are clockwise polarized:

$$\mathbf{E}^I \times \mathbf{B}^I = \left| \begin{array}{ccc} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ n_1^2 - 1 & n_1n_2 & n_1n_3 \\ 0 & -n_3 & n_2 \end{array} \right| = (1 - n_1^2)\ (n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_1)\ \sim \mathbf{k}\ ,$$

$$\mathbf{E}^{II} \times \mathbf{B}^{II} = \left| \begin{array}{ccc} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ n_2n_1 & n_2n_2 - 1 & n_2n_3 \\ n_3 & 0 & -n_1 \end{array} \right| = (1 - n_2^2)\ (n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_1)\ \sim \mathbf{k}\ . \tag{17}$$

Besides, they are independent solutions (but not orthogonal ones)

$$\mathbf{E}^I\mathbf{E}^{II} = -n_1n_2\ , \qquad \mathbf{B}^I\mathbf{B}^{II} = -n_1n_2\ . \tag{18}$$

5 Dual symmetry of Maxwell equations

Let us consider the known dual symmetry in matrix formalism:

$$(-i\partial_0 + \alpha^j \partial_j) \left| \begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array} \right| = 0 \; .$$

It is evident that there exists a simple transform, multiplication by imaginary i, with the following properties:

$$\Psi^D = i\,\Psi \; , \qquad \Psi = -i\,\Psi^D \; , \qquad (-i\partial_0 + \alpha^j \partial_j)\,\Psi^D = 0 \; ,$$
$$\left| \begin{array}{c} 0 \\ \mathbf{E}^D + ic\mathbf{B}^D \end{array} \right| = \left| \begin{array}{c} 0 \\ i\mathbf{E} - c\mathbf{B} \end{array} \right| , \qquad \mathbf{E}^D = -c\mathbf{B} \; , \qquad c\mathbf{B}^D = +\mathbf{E} \; . \tag{19}$$

It is the dual transformation of the electromagnetic field. Several points should be clarified. First, this transformation is not a symmetry operation in presence of external sources. In this case we have

$$(-i\partial_0 + \alpha^j \partial_j) \left| \begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array} \right| = \frac{1}{\epsilon_0} \left| \begin{array}{c} \rho \\ i\mathbf{j} \end{array} \right| \tag{20}$$

and further

$$\Psi^D = i\,\Psi \; , \qquad \Psi = -i\,\Psi^D \; , \qquad (-i\partial_0 + \alpha^j \partial_j)\,\Psi^D = \frac{1}{\epsilon_0} \left| \begin{array}{c} i\rho \\ -\mathbf{j} \end{array} \right| ,$$
$$\left| \begin{array}{c} 0 \\ \mathbf{E}^D + ic\mathbf{B}^D \end{array} \right| = \left| \begin{array}{c} 0 \\ i\mathbf{E} - c\mathbf{B} \end{array} \right| , \qquad \mathbf{E}^D = -c\mathbf{B} \; , \qquad c\mathbf{B}^D = +\mathbf{E} \; . \tag{21}$$

Second, to save the situation one can extend the Maxwell equations by introducing magnetic sources:

$$(-i\partial_0 + \alpha^j \partial_j) \left| \begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array} \right| = \frac{1}{\epsilon_0} \left| \begin{array}{c} \rho_e + i\rho_m \\ i\mathbf{j}_e + \mathbf{j}_m \end{array} \right| , \tag{22}$$

which permits us to consider the dual transformation as a symmetry:

$$\Psi^D = i\,\Psi \; , \qquad \Psi = -i\,\Psi^D \; , \qquad (-i\partial_0 + \alpha^j \partial_j)\,\Psi^D = \frac{1}{\epsilon_0} \left| \begin{array}{c} -\rho_m + i\rho_e \\ i\mathbf{j}_m - \mathbf{j}_e \end{array} \right| ,$$
$$\left| \begin{array}{c} 0 \\ \mathbf{E}^D + ic\mathbf{B}^D \end{array} \right| = \left| \begin{array}{c} 0 \\ i\mathbf{E} - c\mathbf{B} \end{array} \right| , \qquad \mathbf{E}^D = -c\mathbf{B} \; , \qquad c\mathbf{B}^D = +\mathbf{E} \; ,$$
$$\rho_e^D = -\rho_m, \qquad \mathbf{j}_e^D = +\mathbf{j}_m \; , \qquad \rho_m^D = +\rho_e \; , \qquad \mathbf{j}_m^D = -\mathbf{j}_e \; . \tag{23}$$

In real form eqs. (23) will look

$$\partial_0 \frac{\Psi - \Psi^*}{2i} + \alpha^j \partial_j \frac{\Psi + \Psi^*}{2} = \frac{J + J^*}{2} \; ,$$
$$\partial_0 \frac{\Psi + \Psi^*}{2} + \alpha^j \partial_j \frac{\Psi - \Psi^*}{2i} = \frac{J - J^*}{2i} \; ;$$
$$Re(J) = \frac{1}{\epsilon_0} \left| \begin{array}{c} \rho_e \\ \mathbf{j}_m \end{array} \right| , \qquad Im(J) = \frac{1}{\epsilon_0} \left| \begin{array}{c} \rho_m \\ \mathbf{j}_e \end{array} \right| ,$$

that is

$$\partial_0 B + \alpha^j \partial_j E = \text{Re}\,(J)\ , \qquad \partial_0 E + \alpha^j \partial_j B = \text{Im}\,(J)\ . \tag{24}$$

Eqs. (24) in vector notation read

$$\begin{aligned} \text{div}\ \mathbf{E} &= \frac{\rho_e}{\epsilon_0}, \qquad \text{rot}\ \mathbf{E} = +\frac{\mathbf{j}_m}{\epsilon_0} - \frac{\partial c\mathbf{B}}{\partial ct}\ , \\ \text{div}\ c\mathbf{B} &= \frac{\rho_m}{\epsilon_0}\ , \qquad \text{rot}\ c\mathbf{B} = \frac{\mathbf{j}_e}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial ct}\ . \end{aligned} \tag{25}$$

Let us turn again to the Maxwell equation without sources and examine the action of this transformation on a plane electromagnetic wave along the axis z (see (9)):

$$I \qquad E^2 = -k_3 A\ \cos(k_0 x_0 - k_3 x_3)\ , \qquad B^1 = \frac{k_0}{c} A\ \cos(k_0 x_0 - k_3 x_3)\ .$$

After the dual transformation it becomes

$$E^1_D = -k_0 A\ \cos(k_0 x_0 - k_3 x_3)\ , \qquad cB^2_D = -k_3 A\ \cos(k_0 x_0 - k_3 x_3)\ ,$$

It should be noted that the dual solution coincides with the wave of the type II according to (9):

$$II \qquad E^1 = k_3 A\ \cos(k_0 x_0 - k_3 x_3)\ , \qquad B^2 = \frac{k_0}{c} A\ \cos(k_0 x_0 - k_3 x_3)\ .$$

In other words, the dual symmetry provides us with the possibility to construct a new linearly independent solution on the base of the known one.

One addition should be made: the well-known continuous dual symmetry looks as a phase transformation over complex variables:

$$e^{i\chi}\ (\mathbf{E} + ic\ \mathbf{B})\ , \qquad e^{-i\chi}\ (i\ \mathbf{j}_e + \mathbf{j}_m)\ , \qquad e^{-i\chi}\ (\rho_e + i\ \rho_m)\ . \tag{26}$$

6 On separating physical solutions of the Maxwell equations (real-valued scalar function Φ)

Let us consider four formal solutions of the Maxwell equations

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} = \left| \begin{array}{rrrr} iF_0 & F_1 & F_2 & F_3 \\ -F_1 & iF_0 & -F_3 & F_2 \\ -F_2 & F_3 & iF_0 & -F_1 \\ -F_3 & -F_2 & F_1 & iF_0 \end{array} \right|\ , \qquad F_a(x) = \partial_a \Phi(x)\ . \tag{27}$$

Physical solutions should be associated with the following structure:

$$\left| \begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array} \right|$$

when zero-component of the 4×1 column vanishes.

Let the function $\Phi(x)$ be taken as real-valued. We should find all possible solutions to the following equation:

$$\lambda_0 \Psi^0 + \lambda_1 \Psi^1 + \lambda_2 \Psi^2 + \lambda_3 \Psi^3 = \left| \begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array} \right|. \tag{28}$$

Let us separate real and imaginary parts in $\lambda_c: \ \lambda_c = a_c + ib_c$. Relation (28) takes the form

$$
\begin{gathered}
(a_0+ib_0)iF_0 + (a_1+ib_1)F_1 + (a_2+ib_2)F_2 + (a_3+ib_3)F_3 = 0 \ , \\
-(a_0+ib_0)F_1 + (a_1+ib_1)iF_0 - (a_2+ib_2)F_3 + (a_3+ib_3)F_2 = E_1 + icB_1 \ , \\
-(a_0+ib_0)F_2 + (a_1+ib_1)F_3 + (a_2+ib_2)iF_0 - (a_3+ib_3)F_1 = E_2 + icB_2 \ , \\
-(a_0+ib_0)F_3 - (a_1+ib_1)F_2 + (a_2+ib_2)F_1 + (a_3+ib_3)iF_0 = E_3 + icB_3 \ ,
\end{gathered}
$$

from whence it follows

$$
\begin{array}{ll}
-b_0F_0 + a_1F_1 + a_2F_2 + a_3F_3 = 0 \ , & a_0F_0 + b_1F_1 + b_2F_2 + b_3F_3 = 0 \ , \\
-a_0F_1 - b_1F_0 - a_2F_3 + a_3F_2 = E_1 \ , & -b_0F_1 + a_1F_0 - b_2F_3 + b_3F_2 = cB_1 \ , \\
-a_0F_2 + a_1F_3 - b_2F_0 - a_3F_1 = E_2 \ , & -b_0F_2 + b_1F_3 + a_2F_0 - b_3F_1 = cB_2 \ , \\
-a_0F_3 - a_1F_2 + a_2F_1 - b_3F_0 = E_3 \ , & -b_0F_3 - b_1F_2 + b_2F_1 + a_3F_0 = cB_3 \ .
\end{array} \tag{29}
$$

Let us consider equation rot $\mathbf{E} = -\partial_0 c\mathbf{B}$. Taking two identities

$$
\begin{aligned}
\partial_1 E_2 - \partial_2 E_1 = \partial_1[-a_0F_2 + a_1F_3 - b_2F_0 - a_3F_1] \ - \\
-\partial_2[-a_0F_1 - b_1F_0 - a_2F_3 + a_3F_2] \ = \\
= a_1\partial_1F_3 - b_2\partial_1F_0 - a_3\partial_1F_1 + b_1\partial_2F_0 + a_2\partial_2F_3 - a_3\partial_2F_2 \ , \\
-\partial_0 cB_3 = b_0\partial_0F_3 + b_1\partial_0F_2 - b_2\partial_0F_1 - a_3\partial_0F_0 \ ;
\end{aligned}
$$

we produce an equation

$$
\begin{aligned}
a_1\partial_1F_3 - b_2\partial_1F_0 - a_3\partial_1F_1 + b_1\partial_2F_0 + a_2\partial_2F_3 - a_3\partial_2F_2 = \\
= b_0\partial_0F_3 + b_1\partial_0F_2 - b_2\partial_0F_1 - a_3\partial_0F_0 \ ,
\end{aligned}
$$

from whence substituting identity $\partial_0F_0 = \partial_1F_1 + \partial_2F_2 + \partial_3F_3$, we obtain

$$
a_1\partial_1F_3 + a_2\partial_2F_3 = b_0\partial_0F_3 - a_3\partial_3F_3 \ ;
$$

that is

$$
[\ b_0\partial_0 - (a_1\partial_1 + a_2\partial_2 + a_3\partial_3)\ ]\ F_3 = 0 \ . \tag{30}
$$

In the same manner we get

$$
\begin{gathered}
[\ b_0\partial_0 - (a_1\partial_1 + a_2\partial_2 + a_3\partial_3)\ ]\ F_1 = 0 \ , \\
[\ b_0\partial_0 - (a_1\partial_1 + a_2\partial_2 + a_3\partial_3)\ ]\ F_2 = 0 \ .
\end{gathered} \tag{31}
$$

Now consider equation rot $c\mathbf{B} = \partial_0\mathbf{E}$. Calculating two rerms

$$
\begin{aligned}
\partial_1 cB_2 - \partial_2 cB_1 = \\
= b_1\partial_1F_3 + a_2\partial_1F_0 - b_3\partial_1F_1 - a_1\partial_2F_0 + b_2\partial_2F_3 - b_3\partial_2F_2 \ , \\
\partial_0E_3 = -a_0\partial_0F_3 - a_1\partial_0F_2 + a_2\partial_0F_1 - b_3\partial_0F_0 = \\
= -a_0\partial_0F_3 - a_1\partial_0F_2 + a_2\partial_0F_1 - b_3\partial_1F_1 - b_3\partial_2F_2 - b_3\partial_3F_3 \ ,
\end{aligned}
$$

we arrive at

$$
\begin{aligned}
b_1\partial_1F_3 + a_2\partial_1F_0 - b_3\partial_1F_1 - a_1\partial_2F_0 + b_2\partial_2F_3 - b_3\partial_2F_2 = \\
= -a_0\partial_0F_3 - a_1\partial_0F_2 + a_2\partial_0F_1 - b_3\partial_1F_1 - b_3\partial_2F_2 - b_3\partial_3F_3 \ ,
\end{aligned}
$$

or

$$[\, a_0\partial_0 + (b_1\partial_1 + b_2\partial_2 + b_3\partial_3)\,]\, F_3 = 0\ . \tag{32}$$

Analogously, we get

$$\begin{aligned} &[\, a_0\partial_0 + (b_1\partial_1 + b_2\partial_2 + b_3\partial_3)\,]\, F_1 = 0\ ,\\ &[\, a_0\partial_0 + (b_1\partial_1 + b_2\partial_2 + b_3\partial_3)\,]\, F_2 = 0\ . \end{aligned} \tag{33}$$

Now let us consider equation div $\mathbf{E} = 0$:

$$0 = \partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = -a_0\partial_1 F_1 - b_1\partial_1 F_0 - a_2\partial_1 F_3 + a_3\partial_1 F_2 -$$
$$-a_0\partial_2 F_2 + a_1\partial_2 F_3 - b_2\partial_2 F_0 - a_3\partial_2 F_1 - a_0\partial_3 F_3 - a_1\partial_3 F_2 + a_2\partial_3 F_1 - b_3\partial_3 F_0\ ,$$

that is

$$-\, a_0(\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3) - (b_1\partial_1 + b_2\partial_2 + b_3\partial_3) F_0 = 0\ , \tag{34}$$

which is equivalent to

$$[\, a_0\partial_0 + (b_1\partial_1 + b_2\partial_2 + b_3\partial_3)\,]\, F_0 = 0\ . \tag{35}$$

It remains to consider equation div $\mathbf{B} = 0$:

$$0 = -b_0\partial_1 F_1 + a_1\partial_1 F_0 - b_2\partial_1 F_3 + b_3\partial_1 F_2 - b_0\partial_2 F_2 + b_1\partial_2 F_3 + a_2\partial_2 F_0 - b_3\partial_2 F_1 -$$
$$-b_0\partial_3 F_3 - b_1\partial_3 F_2 + b_2\partial_3 F_1 + a_3\partial_3 F_0\ ,$$

or

$$-\, b_0(\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3) + (a_1\partial_1 + a_2\partial_2 + a_3\partial_3) F_0 = 0\ ,$$

which is equivalent to

$$[b_0\partial_0 - (a_1\partial_1 + a_2\partial_2 + a_3\partial_3)]\, F_0 = 0\ . \tag{36}$$

Thus, to construct physical solutions of the Maxwell equations as linear combinations from non-physical ones (see (28))

$$(a_0 + ib_0)\Psi^0 + (a_1 + ib_1)\Psi^1 + (a_2 + ib_2)\Psi^2 + (a_3 + ib_3)\Psi^3 = \left|\begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array}\right| \tag{37}$$

one must satisfy the following 8 equations:

$$\begin{aligned} &[\, b_0\partial_0 - (a_1\partial_1 + a_2\partial_2 + a_3\partial_3)\,]\, F_c = 0\ ,\\ &[\, a_0\partial_0 + (b_1\partial_1 + b_2\partial_2 + b_3\partial_3)\,]\, F_c = 0\ ; \end{aligned} \tag{38}$$

where $c = 0, 1, 2, 3;\ F_c = \partial_c \Phi$.

7 On separating physical solutions of the Maxwell equations (complex-valued scalar function Φ)

Let us consider four formal solutions of the Maxwell equations

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} = \left| \begin{array}{cccc} iF_0 & F_1 & F_2 & F_3 \\ -F_1 & iF_0 & -F_3 & F_2 \\ -F_2 & F_3 & iF_0 & -F_1 \\ -F_3 & -F_2 & F_1 & iF_0 \end{array} \right| , \qquad F_a(x) = \partial_a \Phi(x) \; . \tag{39}$$

Let the function $\Phi(x)$ be taken as complex-valued. We should examine relationship for λ_a defining all possible solutions to the following equation:

$$\lambda_0 \Psi^0 + \lambda_1 \Psi^1 + \lambda_2 \Psi^2 + \lambda_3 \Psi^3 = \left| \begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array} \right| . \tag{40}$$

Let us separate real and imaginary parts in λ_c and $\Phi(x)$ and $F_c(x)$:

$$\lambda_c = a_c + i\, b_c \; , \qquad \Phi(x) = L(x) + i\, K(x) \; , \qquad F_c(x) = L_c(x) + i\, K_c(x) \; .$$

Relation (40) takes the form

$$(a_0 + ib_0) i (L_0 + iK_0) + (a_1 + ib_1)(L_1 + iK_1) +$$
$$+ (a_2 + ib_2)(L_2 + iK_2) + (a_3 + ib_3)(L_3 + iK_3) = 0 \; ,$$

$$-(a_0 + ib_0)(L_1 + iK_1) + (a_1 + ib_1) i (L_0 + iK_0) -$$
$$-(a_2 + ib_2)(L_3 + iK_3) + (a_3 + ib_3)(L_2 + iK_2) = E_1 + icB_1 \; ,$$

$$-(a_0 + ib_0)(L_2 + iK_2) + (a_1 + ib_1)(L_3 + iK_3) +$$
$$+(a_2 + ib_2) i (L_0 + iK_0) - (a_3 + ib_3)(L_1 + iK_1) = E_2 + icB_2 \; ,$$

$$-(a_0 + ib_0)(L_3 + iK_3) - (a_1 + ib_1)(L_2 + iK_2) +$$
$$+(a_2 + ib_2)(L_1 + iK_1) + (a_3 + ib_3) i (L_0 + iK_0) = E_3 + icB_3 \; ,$$

from whence it follows

$$0 = (-b_0 L_0 + a_1 L_1 + a_2 L_2 + a_3 L_3) + (-a_0 K_0 - b_1 K_1 - b_2 K_2 - b_3 K_3) \; ,$$
$$0 = (+a_0 L_0 + b_1 L_1 + b_2 L_2 + b_3 L_3) + (-b_0 K_0 + a_1 K_1 + a_2 K_2 + a_3 K_3) \; ,$$

$$E_1 = (-a_0 L_1 - b_1 L_0 - a_2 L_3 + a_3 L_2) + (+b_0 K_1 - a_1 K_0 + b_2 K_3 - b_3 K_2) \; ,$$
$$B_1 = (-b_0 L_1 + a_1 L_0 - b_2 L_3 + b_3 L_2) + (-a_0 K_1 - b_1 K_0 - a_2 K_3 + a_3 K_2) \; ,$$

$$E_2 = (-a_0 L_2 - b_2 L_0 + a_1 L_3 - a_3 L_1) + (+b_0 K_2 - a_2 K_0 - b_1 K_3 + b_3 K_1) \; ,$$
$$B_2 = (-b_0 L_2 + a_2 L_0 + b_1 L_3 - b_3 L_1) + (-a_0 K_2 - b_2 K_0 + a_1 K_3 - a_3 K_1) \; ,$$

$$E_3 = (-a_0 L_3 - b_3 L_0 - a_1 L_2 + a_2 L_1) + (+b_0 K_3 - a_3 K_0 + b_1 K_2 - b_2 K_1) \; ,$$
$$B_3 = (-b_0 L_3 + a_3 L_0 - b_1 L_2 + b_2 L_1) + (-a_0 K_3 - b_3 K_0 - a_1 K_2 + a_2 K_1) \; , \tag{41}$$

Substituting these expression into Maxwell equations and performing calculation like in previous section we get:

$$
\begin{array}{rcl}
\mathrm{div}\,\mathbf{E} = 0 & \Longrightarrow & -(a_0\partial_0 + b_j\partial_j)\,L_0 + (b_0\partial_0 - a_j\partial_j)\,K_0 = 0\,, \\
\mathrm{div}\,\mathbf{B} = 0 & \Longrightarrow & +(b_0\partial_0 - a_j\partial_j)\,L_0 + (a_0\partial_0 + b_j\partial_j)\,K_0 = 0\,, \\
\partial_2 B_3 - \partial_3 B_2 = +\partial_0 E_1 & \Longrightarrow & -(a_0\partial_0 + b_j\partial_j)\,L_1 + (b_0\partial_0 - a_j\partial_j)\,K_1 = 0\,, \\
\partial_2 E_3 - \partial_3 E_2 = -\partial_0 B_1 & \Longrightarrow & +(b_0\partial_0 - a_j\partial_j)\,L_1 + (a_0\partial_0 + b_j\partial_j)\,K_1 = 0\,, \\
\partial_3 B_1 - \partial_1 B_3 = +\partial_0 E_2 & \Longrightarrow & -(a_0\partial_0 + b_j\partial_j)\,L_2 + (b_0\partial_0 - a_j\partial_j)\,K_2 = 0\,, \\
\partial_3 E_1 - \partial_1 E_3 = -\partial_0 B_2 & \Longrightarrow & +(b_0\partial_0 - a_j\partial_j)\,L_2 + (a_0\partial_0 + b_j\partial_j)\,K_2 = 0\,, \\
\partial_1 B_2 - \partial_2 B_1 = +\partial_0 E_3 & \Longrightarrow & -(a_0\partial_0 + b_j\partial_j)\,L_3 + (b_0\partial_0 - a_j\partial_j)\,K_3 = 0\,, \\
\partial_1 E_2 - \partial_2 E_1 = -\partial_0 B_3 & \Longrightarrow & +(b_0\partial_0 - a_j\partial_j)\,L_3 + (a_0\partial_0 + b_j\partial_j)\,K_3 = 0\,,
\end{array}
\tag{42}
$$

Thus, for physical Maxwell solutions the following equations must hold:

$$
\begin{array}{c}
-\,(a_0\partial_0 + b_j\partial_j)\,L_c + (b_0\partial_0 - a_j\partial_j)\,K_c = 0\,; \\
+(b_0\partial_0 - a_j\partial_j)\,L_c + (a_0\partial_0 + b_j\partial_j)\,K_c = 0\,.
\end{array}
\tag{43}
$$

In particular, we have two more simple equations when taking real or imaginary scalar functions:

$$
\begin{array}{l}
\Phi = L + i\,0\,, \\
\qquad a_c, b_c \;\Longleftarrow\; (a_0\,\partial_0 + b_j\,\partial_j)\,L_c = 0\,,\;\; (b_0\,\partial_0 - a_j\,\partial_j)\,L_c = 0 \\
\Phi = 0 + i\,K \\
\qquad a'_c, b'_c \;\Longleftarrow\; (a'_0\,\partial_0 + b'_j\,\partial_j)\,K_c = 0\,,\;\; (b'_0\,\partial_0 - a'_j\,\partial_j)\,K_c = 0\,.
\end{array}
\tag{44}
$$

In the simplest case of a plane scalar wave

$$
\begin{array}{c}
\Phi = e^{i(k_0 x^0 - k^j x^j)} = \cos\varphi + i\sin\varphi\,, \\
\varphi = k_0\,x^0 - k^j\,x^j = k_c\,x^c\,, \\
L_c + iK_c = -k_c\sin\varphi + i\,k_c\cos\varphi
\end{array}
\tag{45}
$$

previous equations take the form

$$
\begin{array}{l}
\Phi = L + i\,0\,, \\
\qquad a_c, b_c \;\Longleftarrow\; (a_0\,k_0 + b_j\,k_j)\,k_c = 0\,,\;\; (b_0\,k_0 - a_j\,k_j)\,k_c = 0 \\
\Phi = 0 + i\,K \\
\qquad a'_c, b'_c \;\Longleftarrow\; (a'_0\,k_0 + b'_j\,k_j)\,k_c = 0\,,\;\; (b'_0\,k_0 - a'_j\,k_j)\,k_c = 0\,.
\end{array}
\tag{46}
$$

that is

$$
\begin{array}{c}
a'_c = a_c\,, \qquad b'_c = b_c\,, \\
(a_0\,k_0 + b_j\,k_j)\,k_c = 0\,,\;\; (b_0\,k_0 - a_j\,k_j)\,k_c = 0
\end{array}
\tag{47}
$$

In general case (43) equations for a_c, b_c and a'_c, b'_c may not coincide.

Let us turn again to eqs. (44) and translate them to variables

$$L_c = \frac{F_c^* + F_c}{2} \; , \qquad K_c = +i \; \frac{F_c^* - F_c}{2} \; ,$$

$$- \, (a_0 \partial_0 + b_j \partial_j) \; \frac{F_c^* + F_c}{2} + (b_0 \partial_0 - a_j \partial_j) \; i \; \frac{F_c^* - F_c}{2} = 0 \; ;$$
$$+ (b_0 \partial_0 - a_j \partial_j) \; \frac{F_c^* + F_c}{2} + (a_0 \partial_0 + b_j \partial_j) \; i \; \frac{F_c^* - F_c}{2} = 0 \; .$$

or

$$\frac{1}{2} \left[\, -(a_0 \partial_0 + b_j \partial_j) - i \; (b_0 \partial_0 - a_j \partial_j) \, \right] F_c + \frac{1}{2} \left[\, -(a_0 \partial_0 + b_j \partial_j) + i \; (b_0 \partial_0 - a_j \partial_j) \, \right] F_c^* = 0$$
$$\frac{1}{2} \left[\, +(b_0 \partial_0 - a_j \partial_j) - i \; (a_0 \partial_0 + b_j \partial_j) \, \right] F_c + \frac{1}{2} \left[\, +(b_0 \partial_0 - a_j \partial_j) + i \; (a_0 \partial_0 + b_j \partial_j) \, \right] F_c^* = 0 \; ;$$

or

$$\frac{1}{2} \left[\, -(a_0 \partial_0 + b_j \partial_j) - i \; (b_0 \partial_0 - a_j \partial_j) \, \right] F_c + \frac{1}{2} \left[\, -(a_0 \partial_0 + b_j \partial_j) + i \; (b_0 \partial_0 - a_j \partial_j) \, \right] F_c^* = 0$$
$$\frac{1}{2} \left[\, +i \; (b_0 \partial_0 - a_j \partial_j) + (a_0 \partial_0 + b_j \partial_j) \, \right] F_c + \frac{1}{2} \left[\, +i \; (b_0 \partial_0 - a_j \partial_j) - (a_0 \partial_0 + b_j \partial_j) \, \right] F_c^* = 0 \; ;$$

Summing and subtracting two relations, we arrive at

$$\begin{aligned} \left[\, -(a_0 \partial_0 + b_j \partial_j) + i \; (b_0 \partial_0 - a_j \partial_j) \, \right] F_c^* = 0 \\ \left[\, -(a_0 \partial_0 + b_j \partial_j) - i \; (b_0 \partial_0 - a_j \partial_j) \, \right] F_c = 0 \end{aligned} \tag{48}$$

They can be rewritten as follows:

$$\left[\, - \, (a_0 + i b_0) \; \partial_0 + i (a_j + i b_j) \; \partial_j \, \right] F_c = 0 \; ,$$
$$\left[\, - \, (a_0 - i b_0) \; \partial_0 - i (a_j - i b_j) \; \partial_j \, \right] F_c^* = 0 \; ,$$

or shorter

$$\begin{aligned} \left[\, - \, \lambda_0 \; \partial_0 + i \; \lambda_j \; \partial_j \, \right] F_c = 0 \; , \\ \left[\, - \, \lambda_0 \; \partial_0^* - i \; \lambda_j^* \; \partial_j \, \right] F_c^* = 0 \; , \end{aligned} \tag{49}$$

8 Separating physical solutions of the plane wave type

Let us apply the general relations (38) to the case when

$$\Phi = A \; \sin \varphi \; , \qquad \varphi = k_0 x_0 - k_3 x_3 \; ,$$
$$F_0 = k_0 A \cos \varphi \; , \quad F_1 = 0 \; , \quad F_2 = 0 \; , \; F_3 = -k_3 A \cos \varphi \; .$$

Eqs. (38) then give

$$b_0 k_0 + a_3 k_3 = 0 \; , \qquad a_0 k_0 - b_3 k_3 = 0 \; . \tag{50}$$

For a wave spreading in the positive direction $k_3 = +k_0 > 0$, and eqs. (50) give

$$b_0 = -a_3 \;, \qquad b_3 = a_0 \;,$$

and correspondingly relationship (37) looks

$$(a_0 - ia_3)\Psi^0 + (a_3 + ia_0)\Psi^3 + (a_1 + ib_1)\Psi^1 + (a_2 + ib_2)\Psi^2 = \left| \begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array} \right| \;; \tag{51}$$

coefficients at Ψ^1 and Ψ^2 are arbitrary. To understand this fact let us recall the explicit form of Ψ^a – see (8):

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} = A \left| \begin{array}{rrrr} ik_0 & 0 & 0 & -k_0 \\ 0 & ik_0 & k_0 & 0 \\ 0 & -k_0 & ik_0 & 0 \\ k_0 & 0 & 0 & ik_0 \end{array} \right| \cos\varphi \;.$$

One may separate two subsets of non-physical solutions :

$$(a_0 - ia_3)\Psi^0 + (a_3 + ia_0)\Psi^3 = (a_0 - ia_3)k_0 \left\{ \left| \begin{array}{r} i \\ 0 \\ 0 \\ +1 \end{array} \right| + i \left| \begin{array}{r} -1 \\ 0 \\ 0 \\ i \end{array} \right| \right\} \equiv \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right| \;, \tag{52}$$

so relationship (51) reduces to

$$(a_1 + ib_1)\Psi^1 + (a_2 + ib_2)\Psi^2 = (a_1 + ib_1) \left| \begin{array}{r} 0 \\ ik_0 \\ -k_3 \\ 0 \end{array} \right| + (a_2 + ib_2) \left| \begin{array}{r} 0 \\ k_3 \\ ik_3 \\ 0 \end{array} \right| \;. \tag{53}$$

Now let us consider a more general case when

$$\begin{gathered} \Phi(x) = A\sin(k_0 x_0 - \mathbf{kx}) = A\;\sin\varphi \;, \\ F_0 = k_0 A\cos\varphi \;, \qquad F_1 = -k_1 A\cos\varphi \;, \\ F_2 = -k_2 A\cos\varphi \;, \qquad F_3 = -k_3 A\cos\varphi \;. \end{gathered} \tag{54}$$

Eqs. (38) give

$$b_0 k_0 + a_1 k_1 + a_2 k_2 + a_3 k_3 = 0 \;, \qquad a_0 k_0 - b_1 k_1 - b_2 k_2 - b_3 k_3 = 0 \;, \tag{55}$$

and additionally the identity $k_0 = +\sqrt{k_1^2 + k_2^2 + k_3^2}$ holds. One may introduce parametrization $k_j = k_0 n_j ,\; n_j n_j = 1$, then eqs. (55) read

$$b_0 = -(a_1 n_1 + a_2 n_2 + a_3 n_3) \;, \qquad a_0 = (b_1 n_1 + b_2 n_2 + b_3 n_3) \;. \tag{56}$$

Now turning to (37) and excluding variables a_0, b_0 one gets

$$\begin{gathered} [(b_1 n_1 + b_2 n_2 + b_3 n_3) - i(a_1 n_1 + a_2 n_2 + a_3 n_3)]\Psi^0 + \\ +(a_1 + ib_1)\Psi^1 + (a_2 + ib_2)\Psi^2 + (a_3 + ib_3)\Psi^3 = \Psi \;. \end{gathered} \tag{57}$$

Taking into account (see (27)

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} = \left| \begin{array}{rrrr} i & -n_1 & -n_2 & -n_3 \\ n_1 & i & n_3 & -n_2 \\ n_2 & -n_3 & i & n_1 \\ n_3 & n_2 & -n_1 & i \end{array} \right| k_0 A\cos\varphi \;,$$

from (57) we get (factor $k_0 A\cos\varphi$ is omitted)

$$\Psi = \left| \begin{array}{l} [(b_1 - ia_1)n_1 + (b_2 - ia_2)n_2 + (b_3 - ia_3)n_3]\; i - (a_1 + ib_1)n_1 - (a_2 + ib_2)n_2 - (a_3 + ib_3)n_3 \\ [(b_1 - ia_1)n_1 + (b_2 - ia_2)n_2 + (b_3 - ia_3)n_3]\; n_1 + (a_1 + ib_1)i + (a_2 + ib_2)n_3 - (a_3 + ib_3)n_2 \\ [(b_1 - ia_1)n_1 + (b_2 - ia_2)n_2 + (b_3 - ia_3)n_3]\; n_2 - (a_1 + ib_1)n_3 + (a_2 + ib_2)i + (a_3 + ib_3)n_1 \\ [(b_1 - ia_1)n_1 + (b_2 - ia_2)n_2 + (b_3 - ia_3)n_3]\; n_3 + (a_1 + ib_1)n_2 - (a_2 + ib_2)n_1 + (a_3 + ib_3)i \end{array} \right|$$

and further

$$\Psi = \left| \begin{array}{c} 0 \\ [(b_1 - ia_1)n_1 + (b_2 - ia_2)n_2 + (b_3 - ia_3)n_3]\; n_1 + (a_1 + ib_1)i + (a_2 + ib_2)n_3 - (a_3 + ib_3)n_2 \\ [(b_1 - ia_1)n_1 + (b_2 - ia_2)n_2 + (b_3 - ia_3)n_3]\; n_2 - (a_1 + ib_1)n_3 + (a_2 + ib_2)i + (a_3 + ib_3)n_1 \\ [(b_1 - ia_1)n_1 + (b_2 - ia_2)n_2 + (b_3 - ia_3)n_3]\; n_3 + (a_1 + ib_1)n_2 - (a_2 + ib_2)n_1 + (a_3 + ib_3)i \end{array} \right| . \tag{58}$$

Let us introduce three elementary solutions, associated with coefficients $(a_j + ib_j)$:

$$\Psi_{(1)} = \left| \begin{array}{l} 0 \\ (b_1 - ia_1)\; n_1 n_1 + (a_1 + ib_1)i \\ (b_1 - ia_1)\; n_1 n_2 - (a_1 + ib_1)n_3 \\ (b_1 - ia_1)\; n_1 n_3 + (a_1 + ib_1)n_2 \end{array} \right| , \mathbf{E}_{(1)} = \left| \begin{array}{l} b_1 n_1 n_1 - b_1 \\ b_1 n_1 n_2 - a_1 n_3 \\ b_1 n_1 n_3 + a_1 n_2 \end{array} \right| , c\mathbf{B}_{(1)} = \left| \begin{array}{l} -a_1 n_1 n_1 + a_1 \\ -a_1 n_1 n_2 - b_1 n_3 \\ -a_1 n_1 n_3 + b_1 n_2 \end{array} \right| ,$$

$$\Psi_{(2)} = \left| \begin{array}{l} 0 \\ (b_2 - ia_2)\; n_2 n_1 + (a_2 + ib_2)n_3 \\ (b_2 - ia_2)\; n_2 n_2 + (a_2 + ib_2)i \\ (b_2 - ia_2)\; n_2 n_3 - (a_2 + ib_2)n_1 \end{array} \right| , \mathbf{E}_{(2)} = \left| \begin{array}{l} b_2 n_2 n_1 + a_2 n_3 \\ b_2 n_2 n_2 - b_2 \\ b_2 n_2 n_3 - a_2 n_1 \end{array} \right| , c\mathbf{B}_{(2)} = \left| \begin{array}{l} -a_2 n_2 n_1 + b_2 n_3 \\ -a_2 n_2 n_2 + a_2 \\ -a_2 n_2 n_3 - b_2 n_1 \end{array} \right| ,$$

$$\Psi_{(3)} = \left| \begin{array}{l} 0 \\ (b_3 - ia_3)\; n_3 n_1 - (a_3 + ib_3)n_2 \\ (b_3 - ia_3)\; n_3 n_2 + (a_3 + ib_3)n_1 \\ (b_3 - ia_3)\; n_3\; n_3 + (a_3 + ib_3)i \end{array} \right| , \mathbf{E}_{(3)} = \left| \begin{array}{l} b_3 n_3 n_1 - a_3 n_2 \\ b_3 n_3 n_2 + a_3 n_1 \\ b_3 n_3\; n_3 - b_3 \end{array} \right| , c\mathbf{B}_{(3)} = \left| \begin{array}{l} -a_3 n_3 n_1 - b_3 n_2 \\ -a_3 n_3 n_2 + b_3 n_1 \\ -a_3 n_3\; n_3 + a_3 \end{array} \right| . \tag{59}$$

Let us show that three types of solutions are linearly dependent. It suffices to examine their linear combinations (for definite consider electric field):

$$A_1\, \mathbf{E}_{(1)} + A_2\, \mathbf{E}_{(2)} + A_3\, \mathbf{E}_{(3)} = 0\, ,$$

that is

$$A_1 \left| \begin{array}{l} b_1 n_1 n_1 - b_1 \\ b_1 n_1 n_2 - a_1 n_3 \\ b_1 n_1 n_3 + a_1 n_2 \end{array} \right| + A_2 \left| \begin{array}{l} b_2 n_2 n_1 + a_2 n_3 \\ b_2 n_2 n_2 - b_2 \\ b_2 n_2 n_3 - a_2 n_1 \end{array} \right| + A_3 \left| \begin{array}{l} b_3 n_3 n_1 - a_3 n_2 \\ b_3 n_3 n_2 + a_3 n_1 \\ b_3 n_3\; n_3 - b_3 \end{array} \right| = \left| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right| .$$

It remains to show that the determinant of the 3×3 matrix vanishes

$$\det \left| \begin{array}{lll} b_1 n_1 n_1 - b_1 & b_2 n_2 n_1 + a_2 n_3 & b_3 n_3 n_1 - a_3 n_2 \\ b_1 n_1 n_2 - a_1 n_3 & b_2 n_2 n_2 - b_2 & b_3 n_3 n_2 + a_3 n_1 \\ b_1 n_1 n_3 + a_1 n_2 & b_2 n_2 n_3 - a_2 n_1 & b_3 n_3\; n_3 - b_3 \end{array} \right| = 0\, . \tag{60}$$

Let $b_1 n_1 = s_1, b_2 n_2 = s_2, b_3 n_3 = s_3$ then eq. (60) looks

$$0 = \left| \begin{array}{lll} s_1 n_1 - b_1 & s_2 n_1 + a_2 n_3 & s_3 n_1 - a_3 n_2 \\ s_1 n_2 - a_1 n_3 & s_2 n_2 - b_2 & s_3 n_2 + a_3 n_1 \\ s_1 n_3 + a_1 n_2 & s_2 n_3 - a_2 n_1 & s_3 n_3 - b_3 \end{array} \right| =$$

$$\begin{array}{c} = (s_1 n_1 - b_1)[(s_2 n_2 - b_2)(s_3 n_3 - b_3) - (s_3 n_2 + a_3 n_1)(s_2 n_3 - a_2 n_1)] - \\ -(s_1 n_2 - a_1 n_3)[(s_2 n_1 + a_2 n_3)(s_3 n_3 - b_3) - (s_3 n_1 - a_3 n_2)(s_2 n_3 - a_2 n_1)] + \\ +(s_1 n_3 + a_1 n_2)[(s_2 n_1 + a_2 n_3)(s_3 n_2 + a_3 n_1) - (s_3 n_1 - a_3 n_2)(s_2 n_2 - b_2)] = \end{array}$$

$$\begin{array}{c} = (s_1 n_1 - b_1)[s_2 s_3 n_2 n_3 - s_2 n_2 b_3 - s_3 n_3 b_2 + b_2 b_3 - s_2 s_3 n_2 n_3 + s_3 n_1 n_2 a_2 - n_1 n_3 a_3 s_2 + a_2 a_3 n_1^2] - \\ -(s_1 n_2 - a_1 n_3)[s_2 s_3 n_1 n_3 - s_2 n_1 b_3 + a_2 s_3 n_3^2 - a_2 n_3 b_3 - s_2 s_3 n_1 n_3 + s_3 a_2 n_1^2 + a_3 s_2 n_2 n_3 - a_3 a_2 n_1 n_2] + \\ +(s_1 n_3 + a_1 n_2)[s_2 s_3 n_1 n_2 + s_2 a_3 n_1^2 + a_2 s_3 n_2 n_3 + a_2 a_3 n_1 n_3 - s_2 s_3 n_1 n_2 + s_3 b_2 n_1 + a_3 s_2 n_2^2 - a_3 b_2 n_2] \end{array}$$

$$\begin{array}{r} = -s_1 n_1 s_2 n_2 b_3 - s_1 n_1 s_3 n_3 b_2 + s_1 n_1 b_2 b_3 + s_1 n_1 s_3 n_1 n_2 a_2 - s_1 n_1 n_1 n_3 a_3 s_2 + s_1 n_1 a_2 a_3 n_1^2 - \\ +b_1 s_2 n_2 b_3 + b_1 s_3 n_3 b_2 - b_1 b_2 b_3 - b_1 s_3 n_1 n_2 a_2 + b_1 n_1 n_3 a_3 s_2 - b_1 a_2 a_3 n_1^2 + \\ +s_1 n_2 s_2 n_1 b_3 - s_1 n_2 a_2 s_3 n_3^2 + s_1 n_2 a_2 n_3 b_3 + s_1 n_2 s_3 a_2 n_1^2 - s_1 n_2 a_3 s_2 n_2 n_3 + s_1 n_2 a_3 a_2 n_1 n_2 - \\ -a_1 n_3 s_2 n_1 b_3 + a_1 n_3 a_2 s_3 n_3^2 - a_1 n_3 a_2 n_3 b_3 + a_1 n_3 s_3 a_2 n_1^2 + a_1 n_3 a_3 s_2 n_2 n_3 - a_1 n_3 a_3 a_2 n_1 n_2 + \\ +s_1 n_3 s_2 a_3 n_1^2 + s_1 n_3 a_2 s_3 n_2 n_3 + s_1 n_3 a_2 a_3 n_1 n_3 + s_1 n_3 s_3 b_2 n_1 + s_1 n_3 a_3 s_2 n_2^2 - s_1 n_3 a_3 b_2 n_2 + \\ +a_1 n_2 s_2 a_3 n_1^2 + a_1 n_2 a_2 s_3 n_2 n_3 + a_1 n_2 a_2 a_3 n_1 n_3 + a_1 n_2 s_3 b_2 n_1 + a_1 n_2 a_3 s_2 n_2^2 - a_1 n_2 a_3 b_2 n_2 \end{array}$$

and further

$$\begin{array}{r} 0 = -b_1 b_2 b_3 n_1^2 n_3^2 + b_1 b_2 b_3 n_1^2 + b_1 b_3 a_2 n_1^3 n_2 n_3 - b_1 b_2 a_3 n_1^3 n_2 n_3 + b_1 a_2 a_3 n_1^4 - \\ +b_1 b_2 b_3 n_2^2 + b_1 b_2 b_3 n_3^2 - b_1 b_2 b_3 - b_1 b_3 a_2 n_1 n_2 n_3 + b_1 b_2 a_3 n_1 n_2 n_3 - b_1 a_2 a_3 n_1^2 + \\ -b_1 b_3 a_2 n_1 n_2 n_3^3 + b_1 b_3 a_2 n_1 n_2 n_3 + b_1 b_3 a_2 n_2 n_3 n_1^3 - b_1 b_2 a_3 n_1 n_2 n_2^2 n_3 + b_1 a_3 a_2 n_1^2 n_2^2 - \\ -b_2 b_3 a_1 n_3 n_2 n_1 + a_1 b_3 a_2 n_3^4 - a_1 b_3 a_2 n_3^2 + b_3 a_1 a_2 n_3^2 n_1^2 + b_2 a_1 a_3 n_2^2 n_3^2 - a_1 a_2 a_3 n_1 n_2 n_3 + \\ +b_1 b_2 a_3 n_2 n_3 n_1^3 + b_1 a_2 b_3 n_3^3 n_1 n_2 + b_1 a_2 a_3 n_1^2 n_3^2 + b_1 b_2 b_3 n_3^2 n_1^2 + b_1 b_2 a_3 n_1 n_3 n_2^3 - b_1 a_3 b_2 n_1 n_2 n_3 + \\ +a_1 b_2 a_3 n_2^2 n_1^2 + a_1 a_2 b_3 n_2^2 n_3^2 + a_1 a_2 a_3 n_1 n_2 n_3 + a_1 b_2 b_3 n_1 n_2 n_3 + a_1 a_3 b_2 n_2^4 - a_1 a_3 b_2 n_2^2 \equiv= 0 \; ; \end{array}$$

it is easily verified that all terms cancel out each other indeed.

Let let us consider one other example and start with a complex scalar plane wave:

$$\Phi(x) = e^{i(k_0 x_0 - k_j x_j)} \; , \qquad L_c = k_c \cos\varphi \; , \quad K_c = k_c \sin\varphi \; ;$$

and eqs. (43) take the form

$$\begin{array}{c} -\,(a_0 k_0 + b_j k_j)\; k_c \cos\varphi + (b_0 k_0 - a_j k_j)\; k_c \sin\varphi = 0 \; ; \\ +(b_0 k_0 - a_j k_j)\; k_c \cos\varphi + (a_0 k_0 + b_j k_j)\; k_c \sin\varphi = 0 \; . \end{array}$$

from whence it follow

$$b_0 k_0 + (a_1 k_1 + a_2 k_2 + a_3 k_3) = 0 \; , \qquad a_0 k_0 - (b_1 k_1 + b_2 k_2 + b_3 k_3 = 0 \; .$$

or

$$b_0 = -(a_1 n_1 + a_2 n_2 + a_3 n_3) \; , \qquad a_0 = +(b_1 n_1 + b_2 n_2 + b_3 n_3) \; . \tag{61}$$

Correspondingly, eq. (37) gives

$$\Psi(x) = [+(b_1 n_1 + b_2 n_2 + b_3 n_3) - i(a_1 n_1 + a_2 n_2 + a_3 n_3)]\ \Psi^0 +$$
$$+(a_1 + ib_1)\Psi^1 + (a_2 + ib_2)\Psi^2 + (a_3 + ib_3)\Psi^3 = \begin{vmatrix} 0 \\ \mathbf{E} + ic\mathbf{B} \end{vmatrix}$$

where (see (27))

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} = \begin{vmatrix} i & -n_1 & -n_2 & -n_3 \\ n_1 & i & n_3 & -n_2 \\ n_2 & -n_3 & i & n_1 \\ n_3 & n_2 & -n_1 & i \end{vmatrix} ik_0 A\ (\cos\varphi + i\sin\varphi),\ .$$

Further we get

$$\Psi = [-i(\mathbf{a} + i\mathbf{b})\ \mathbf{n}\ \Psi^0 + (a_1 + ib_1)\Psi^1 + (a_2 + ib_2)\Psi^2 + (a_3 + ib_3)\Psi^3] =$$
$$= \begin{vmatrix} (\mathbf{a} + i\mathbf{b})\mathbf{n} - (a_1 + ib_1)n_1 - (a_2 + ib_2)n_2 - (a_3 + ib_3)n_3 \\ -i(\mathbf{a} + i\mathbf{b})\mathbf{n}\ n_1 + (a_1 + ib_1)i + (a_2 + ib_2)n_3 - (a_3 + ib_3)n_2 \\ -i(\mathbf{a} + i\mathbf{b})\mathbf{n}\ n_2 - (a_1 + ib_1)n_3 + (a_2 + ib_2)i + (a_3 + ib_3)n_1 \\ -i(\mathbf{a} + i\mathbf{b})\mathbf{n}\ n_3 + (a_1 + ib_1)n_2 - (a_2 + ib_2)n_1 + (a_3 + ib_3)i \end{vmatrix} ik_0 A\ (\cos\varphi + i\sin\varphi),\ .$$

that is

$$\Psi = \begin{vmatrix} 0 \\ (\mathbf{a} + i\mathbf{b})\mathbf{n}n_1 - (a_1 + ib_1) + i(a_2 + ib_2)n_3 - i(a_3 + ib_3)n_2 \\ (\mathbf{a} + i\mathbf{b})\mathbf{n}n_2 - i(a_1 + ib_1)n_3 - (a_2 + ib_2) + i(a_3 + ib_3)n_1 \\ (\mathbf{a} + i\mathbf{b})\mathbf{n}n_3 + i(a_1 + ib_1)n_2 - i(a_2 + ib_2)n_1 - (a_3 + ib_3) \end{vmatrix} k_0 A\ (\cos\varphi + i\sin\varphi)$$

which is equivalent to

$$\Psi = \begin{vmatrix} 0 \\ (\mathbf{an}n_1 - a_1 - b_2 n_3 + b_3 n_2) + i(\mathbf{bn}n_1 - b_1 - n_2 a_3 + n_3 a_2 \\ (\mathbf{an}n_2 - a_2 - b_3 n_1 + b_1 n_3) + i(\mathbf{bn}n_2 - b_2 - n_3 a_1 + n_1 a_2 \\ (\mathbf{an}n_3 - a_3 - b_1 n_2 + b_2 n_1) + i(\mathbf{bn}n_3 - b_3 - n_1 a_2 + n_2 a_1 \end{vmatrix} k_0 A\ (\cos\varphi + i\sin\varphi)\ . \tag{62}$$

With the use of notation

$$\mathbf{L} = \mathbf{n}\ (\mathbf{na}) - \mathbf{a} - \mathbf{b} \times \mathbf{n}\ , \qquad \mathbf{C} = \mathbf{n}\ (\mathbf{nb}) - \mathbf{b} + \mathbf{a} \times \mathbf{n},$$

relationship (62) can be written shorter

$$\mathbf{E} + ic\mathbf{B} = (\mathbf{L} + i\mathbf{C})\ k_0 A\ (\cos\varphi + i\sin\varphi)\ , \tag{63}$$

from whence it follow

$$\mathbf{E} = k_0 A\ (\cos\varphi \mathbf{L} - \sin\varphi \mathbf{C})\ , \qquad c\mathbf{B} = k_0 A\ (\sin\varphi \mathbf{L} + \cos\varphi \mathbf{C})\ . \tag{64}$$

One can readily prove identities:

$$\mathbf{L}^2 = \mathbf{C}^2 = \mathbf{a}^2 + \mathbf{b}^2 - (\mathbf{na})^2 - (\mathbf{nb})^2 + 2\mathbf{n}\ (\mathbf{a} \times \mathbf{b})\ ,$$
$$\mathbf{L}\ \mathbf{C} = 0\ , \qquad \mathbf{EB} = 0\ , \qquad \mathbf{E}^2 = c^2\mathbf{B}^2\ ,$$
$$\mathbf{Ln} = 0\ , \qquad \mathbf{Cn} = 0, \mathbf{En} = 0\ , \qquad \mathbf{Bn} = 0\ . \tag{65}$$

General expressions for $\mathbf{L}, \mathbf{C}$ may be decomposed into the sum:

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 \; , \qquad \mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 \; ,$$

$$I \qquad \underline{\mathbf{b} = 0:} \qquad \mathbf{L}_1 = \mathbf{n}\;(\mathbf{na}) - \mathbf{a}\; , \qquad \mathbf{C}_1 = \mathbf{a} \times \mathbf{n}\; ,$$

$$\mathbf{E}_1 \times c\mathbf{B}_1 = k_0^2 A^2\; (\mathbf{L_1} \times \mathbf{C}_1) = k_0^2 A^2\; [\; a^2 - (\mathbf{na})^2\;]\; \mathbf{n}\; ;$$

$$II \qquad \underline{\mathbf{a} = 0:} \qquad \mathbf{L}_2 = -\mathbf{b} \times \mathbf{n}\; , \qquad \mathbf{C}_2 = \mathbf{n}\;(\mathbf{nb}) - \mathbf{b}\; ,$$

$$\mathbf{E}_2 \times c\mathbf{B}_2 = k_0^2 A^2\; (\mathbf{L_2} \times \mathbf{C}_2) = k_0^2 A^2\; [\; b^2 - (\mathbf{nb})^2\;]\; \mathbf{n}\; , \tag{66}$$

In other words, these two electromagnetic wave are clockwise polarized.

For a particular case when $\mathbf{b} = \mathbf{a}$, we get

$$\begin{aligned}
I \qquad \mathbf{L}_1 = \mathbf{n}\;(\mathbf{na}) - \mathbf{a}\; , &\qquad \mathbf{C}_1 = \mathbf{a} \times \mathbf{n}\; , \\
\mathbf{E}_1 &= k_0 A(\cos\varphi\; \mathbf{L}_1 - \sin\varphi\; \mathbf{C}_1)\; , \\
c\mathbf{B}_1 &= k_0 A\; (\sin\varphi \mathbf{L}_1 + \cos\varphi \mathbf{C}_1)\; ; \\
II \qquad \mathbf{L}_2 = -\mathbf{a} \times \mathbf{n} = -\mathbf{C}_1\; , &\qquad \mathbf{C}_2 = \mathbf{n}\;(\mathbf{na}) - \mathbf{a} = \mathbf{L}_1\; , \\
\mathbf{E}_2 &= k_0 A(-\sin\varphi\; \mathbf{L}_1 - \cos\varphi\; \mathbf{C}_1)\; , \\
c\mathbf{B}_2 &= k_0 A\; (-\sin\varphi \mathbf{C}_1 + \cos\varphi \mathbf{L}_1)\; .
\end{aligned} \tag{67}$$

so constructed waves are linearly independent and orthogonal:

$$\mathbf{E}_1 \mathbf{E}_2 = 0\; , \qquad \mathbf{B}_1 \mathbf{B}_2 = 0\; . \tag{68}$$

In view of linearity of the Maxwell equations any linear combination of the type is a solution as well:

$$\mathbf{E} = c_1\; \mathbf{E}_1 + c_2\; \mathbf{E}_2\; , \qquad \mathbf{B} = c_1\; \mathbf{B}_1 + c_2\; \mathbf{B}_2\; . \tag{69}$$

9 Cylindrical waves

Let us start with a cylindrical scalar wave (below for brevity $k_0 = E$):

$$\Phi = e^{iEx_0}\; e^{ikz}\; e^{im\phi}\; R(\rho)\; , \tag{70}$$

$$\begin{aligned}
& x_1 = \rho\; \cos\phi\; , \; x_2 = \rho\; \sin\phi\; , \; x_3 = z\; , \\
& \frac{\partial}{\partial x_1} = \frac{\partial \rho}{\partial x_1}\frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial x_2}\frac{\partial}{\partial \phi} = \cos\phi\; \frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\; \frac{\partial}{\partial \phi}\; , \\
& \frac{\partial}{\partial x_2} = \frac{\partial \rho}{\partial x_2}\frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial x_2}\frac{\partial}{\partial \phi} = \sin\phi\; \frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\; \frac{\partial}{\partial \phi}\; .
\end{aligned} \tag{71}$$

Corresponding electromagnetic solutions are to be constructed on the base of relations:

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} = \lambda_0 \Psi^0 + \lambda_1 \Psi^1 + \lambda_2 \Psi^2 + \lambda_3 \Psi^3 = \left| \begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array} \right| . \tag{72}$$

Let us find F_a:

$$\begin{aligned}
F_0 = iE\; e^{iEx_0}\; e^{ikz}\; e^{im\phi}\; R(\rho)\; , &\qquad F_3 = ik\; e^{iEx_0}\; e^{ikz}\; e^{im\phi}\; R(\rho)\; , \\
F_1 &= e^{iEx_0}\; e^{ikz}\; e^{im\phi}\; (\cos\phi\; \frac{d}{d\rho} - im\; \frac{\sin\phi}{\rho})\; R\; , \\
F_2 &= e^{iEx_0}\; e^{ikz}\; e^{im\phi}\; (\sin\phi\; \frac{d}{d\rho} + im\; \frac{\cos\phi}{\rho})\; R\; ,
\end{aligned} \tag{73}$$

The main equations to solve are

$$(-\lambda_0\partial_0 + i\lambda_j\partial_j)\ F_c = 0 \tag{74}$$

or

$$[\ -i\lambda_0 E - \lambda_3 k + i\lambda_1(\cos\phi\ \frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\ \frac{\partial}{\partial\phi}) + i\lambda_2(\sin\phi\ \frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\ \frac{\partial}{\partial\phi})\ ]\ F_c = 0\ ,$$

or

$$[\ -i\lambda_0 E - \lambda_3 k + i(\lambda_1\cos\phi + \lambda_2\sin\phi)\ \frac{\partial}{\partial\rho} + i\frac{-\lambda_1\sin\phi + \lambda_2\cos\phi}{\rho}\frac{\partial}{\partial\phi}\ ]\ F_c = 0\ . \tag{75}$$

Let $c = 1$:

$$[-i\lambda_0 E - \lambda_3 k + i(\lambda_1\cos\phi + \lambda_2\sin\phi)\ \frac{\partial}{\partial\rho} +$$
$$+ i\frac{-\lambda_1\sin\phi + \lambda_2\cos\phi}{\rho}\frac{\partial}{\partial\phi}]e^{im\phi}\ (\cos\phi\frac{d}{d\rho} - im\frac{\sin\phi}{\rho})R = 0,$$

Let $c = 2$:

$$[-i\lambda_0 E - \lambda_3 k + i(\lambda_1\cos\phi + \lambda_2\sin\phi)\ \frac{\partial}{\partial\rho} +$$
$$+ i\frac{-\lambda_1\sin\phi + \lambda_2\cos\phi}{\rho}\frac{\partial}{\partial\phi}]e^{im\phi}\ (\sin\phi\frac{d}{d\rho} + im\frac{\cos\phi}{\rho})R = 0,$$

Noting that

$$F_1 + iF_2 = e^{iEx_0}\ e^{ikz}\ e^{i(m+1)\phi}(\frac{d}{d\rho} - \frac{m}{\rho})R\ ,$$
$$F_1 - iF_2 = e^{iEx_0}\ e^{ikz}\ e^{i(m-1)\phi}(\frac{d}{d\rho} + \frac{m}{\rho})R\ , \tag{76}$$

from previous two equations we get

$$[-i\lambda_0 E - \lambda_3 k + i(\lambda_1\cos\phi + \lambda_2\sin\phi)\ \frac{\partial}{\partial\rho} + \frac{\lambda_1\sin\phi - \lambda_2\cos\phi}{\rho}\ (m+1)\ ]\ (\frac{d}{d\rho} - \frac{m}{\rho})R = 0\ ,$$

$$[-i\lambda_0 E - \lambda_3 k + i(\lambda_1\cos\phi + \lambda_2\sin\phi)\ \frac{\partial}{\partial\rho} + \frac{\lambda_1\sin\phi - \lambda_2\cos\phi}{\rho}\ (m-1)\ ]\ (\frac{d}{d\rho} + \frac{m}{\rho})R = 0\ . \tag{77}$$

In turn, when $c = 0, 3$, we get one the same equation:

$$[\ -i\lambda_0 E - \lambda_3 k + i(\lambda_1\cos\phi + \lambda_2\sin\phi)\ \frac{\partial}{\partial\rho} + \frac{\lambda_1\sin\phi - \lambda_2\cos\phi}{\rho}\ m\ ]\ R(\rho) = 0\ , \tag{78}$$

We readily note that there exist very simple way to satisfy these three equations on parameters λ_c:

$$-\ i\lambda_0 E - \lambda_3 k = 0\ , \qquad \lambda_1 = 0\ , \qquad \lambda_2 = 0\ . \tag{79}$$

Let us demonstrate that no other solutions exist. To this end, with notation

$$C = -i\lambda_0 E - \lambda_3 k\ , \qquad A = \lambda_1\cos\phi + \lambda_2\sin\phi\ , \qquad B = \lambda_1\sin\phi - \lambda_2\cos\phi\ .$$

let us rewrite eqs. (77) – (78) in the form

$$[\, C + iA\frac{d}{d\rho} + \frac{B}{\rho}\,(m+1)\,]\,(\frac{dR}{d\rho} - \frac{m}{\rho}\,R) = 0\,,$$

$$[\, C + iA\frac{d}{d\rho} + \frac{B}{\rho}\,(m-1)\,]\,(\frac{dR}{d\rho} + \frac{m}{\rho}\,R) = 0\,,$$

$$[\, C + iA\frac{d}{d\rho} + \frac{B}{\rho}m\,]\,R = 0\,. \tag{80}$$

Combining two first equations in (80)(summing and subtracting), we get

$$C\frac{dR}{d\rho} + iA\,\frac{d^2R}{d\rho^2} + \frac{mB}{\rho}\frac{dR}{d\rho} - \frac{mB}{\rho^2}R = 0\,,$$

$$-\frac{mC}{\rho}R - imA\,\frac{d}{d\rho}\frac{R}{\rho} + \frac{B}{\rho}\frac{dR}{d\rho} - m^2\frac{B}{\rho^2}R = 0\,,$$

$$C\,R + iA\,\frac{dR}{d\rho} + m\,\frac{B}{\rho}\,R = 0\,. \tag{81}$$

After differentiating the third equation will look

$$C\,\frac{dR}{d\rho} + iA\,\frac{d^2R}{d\rho^2} + \frac{mB}{\rho}\frac{dR}{d\rho} - \frac{mB}{\rho^2}\,R = 0$$

which coincides with the first equation in (81). Therefore, the system (81) is equivalent to

$$-\,\frac{mC}{\rho}R - imA\,\frac{d}{d\rho}\frac{R}{\rho} + \frac{B}{\rho}\frac{dR}{d\rho} - m^2\frac{B}{\rho^2}R = 0\,,$$

$$C\,R + iA\,\frac{dR}{d\rho} + m\,\frac{B}{\rho}\,R = 0\,. \tag{82}$$

The system (**??**) can be rewritten as follows:

$$-\,m\,\frac{C}{\rho}\,R + \frac{B}{\rho}\,\frac{dR}{d\rho} + i\,\frac{mA}{\rho^2}\,R - \frac{m}{\rho}\,(iA\,\frac{dR}{d\rho} + m\,\frac{B}{\rho}\,R) = 0\,,$$

$$C\,R + iA\,\frac{dR}{d\rho} + m\,\frac{B}{\rho}\,R = 0\,.$$

here the first equation with the use of the second equation gives

$$\frac{B}{\rho}\,\frac{dR}{d\rho} + i\,\frac{mA}{\rho^2}\,R = 0\,.$$

Therefore system (82) is equivalent to

$$B\,\frac{dR}{d\rho} + i\,\frac{mA}{\rho}\,R = 0\,,$$

$$iA\,\frac{dR}{d\rho} + m\,\frac{B}{\rho}\,R + C\,R = 0$$

which in turn is equivalent to

$$(B + iA)(\frac{d}{d\rho} + \frac{m}{\rho})R + C\,R = 0\,,$$

$$(B - iA)(\frac{d}{d\rho} - \frac{m}{\rho})R - C\,R = 0\,, \tag{83}$$

Noting identity $(A+iB)(A-iB)=\lambda_1^2+\lambda_2^2$, one reduces eqs. (83) to the form

$$(\lambda_1^2+\lambda_2^2)(\frac{d}{d\rho}+\frac{m}{\rho})R+(B-iA)C\;R=0\;,$$
$$(\lambda_1^2+\lambda_2^2)(\frac{d}{d\rho}-\frac{m}{\rho})R+(B+iA)C\;R=0\;. \qquad (84)$$

Remembering that

$$C=-i\lambda_0E-\lambda_3k\;,\qquad A=\lambda_1\cos\phi+\lambda_2\sin\phi\;,\qquad B=\lambda_1\sin\phi-\lambda_2\cos\phi\;.$$

we immediately conclude that eqs. (84) can be satisfied only by the following way:

$$(\lambda_1^2+\lambda_2^2)\;,\qquad C=0\;. \qquad (85)$$

In other words, This means that relations (79) provide us with the only possible solution in terms of two (1×4) columns

$$\Psi=\lambda_0\Psi^0+\lambda_3\Psi^3\;,\qquad \underline{-\lambda_0\;E+i\;\lambda_3k=0}\;,$$
$$\Psi^0=\left|\begin{array}{r} iF_0(x)\\ -F_1(x)\\ -F_2(x)\\ -F_3(x)\end{array}\right|\;,\quad \Psi^3=\left|\begin{array}{r} F_3(x)\\ F_2(x)\\ -F_1(x)\\ iF_0(x)\end{array}\right|\;,$$
$$F_0=iE\;\Phi\;,\qquad F_1=(\cos\phi\;\frac{d}{d\rho}-im\;\frac{\sin\phi}{\rho})\;\Phi\;,$$
$$F_2=(\sin\phi\;\frac{d}{d\rho}+im\;\frac{\cos\phi}{\rho})\;\Phi\;,\;,\qquad F_3=ik\;\Phi\;, \qquad (86)$$

Solution explicitly reads by components:

$$(\Psi)_0=(-\lambda_0\;E+i\;\lambda_3k)\;\Phi=0\;,$$

$$E_3+iB_3=(\Psi)_3=(-\lambda_0\;ik-\lambda_3E)\;\Phi=-\lambda_3\;\frac{E^2-k^2}{E}\;\Phi\;,$$
$$E_1+iB_1=(\Psi)_1=\left[-\lambda_0\;(\cos\phi\;\frac{d}{d\rho}-im\;\frac{\sin\phi}{\rho})\;+\lambda_3\;(\sin\phi\;\frac{d}{d\rho}+im\;\frac{\cos\phi}{\rho})\right]\;\Phi;,$$
$$E_2+iB_2=(\Psi)_2=\left[\;(-\lambda_0\;(\sin\phi\;\frac{d}{d\rho}+im\;\frac{\cos\phi}{\rho})-\lambda_3\;(\cos\phi\;\frac{d}{d\rho}-im\;\frac{\sin\phi}{\rho})\right]\;\Phi\;.$$

After simple rewriting we get

$$E_3+iB_3=(\Psi)_3=-\lambda_3\;\frac{E^2-k^2}{E}\;\Phi\;,$$

$$E_1+iB_1=\frac{\lambda_3}{E}\;\left[-ik\;(\cos\phi\;\frac{d}{d\rho}-im\;\frac{\sin\phi}{\rho})\;+E\;(\sin\phi\;\frac{d}{d\rho}+im\;\frac{\cos\phi}{\rho})\right]\;\Phi\;,$$
$$E_2+iB_2=\frac{\lambda_3}{E}\;\left[\;(-ik\;(\sin\phi\;\frac{d}{d\rho}+im\;\frac{\cos\phi}{\rho})-E\;(\cos\phi\;\frac{d}{d\rho}-im\;\frac{\sin\phi}{\rho})\right]\;\Phi\;.$$

For special cases we get most simplicity:

$$\underline{k=+E}\;,\qquad E_3+iB_3=0\;,$$

$$E_1+iB_1 = \lambda_3 \left[-i\,(\cos\phi\,\frac{d}{d\rho} - im\,\frac{\sin\phi}{\rho})\ + (\sin\phi\,\frac{d}{d\rho} + im\,\frac{\cos\phi}{\rho}) \right] \Phi \ = -i\,\lambda_3\ e^{i\phi}(\frac{d}{d\rho} - \frac{m}{\rho})\Phi\ ,$$

$$E_2+iB_2 = \lambda_3 \left[(-i\,(\sin\phi\,\frac{d}{d\rho} + im\,\frac{\cos\phi}{\rho}) - (\cos\phi\,\frac{d}{d\rho} - im\,\frac{\sin\phi}{\rho}) \right] \Phi = -\lambda_3\ e^{i\phi}(\frac{d}{d\rho} - \frac{m}{\rho})\Phi\ ,\ ;$$

$$\underline{k=-E}\,, \qquad E_3+iB_3 = 0\ ,$$

$$E_1+iB_1 = \lambda_3 \left[+i\,(\cos\phi\,\frac{d}{d\rho} - im\,\frac{\sin\phi}{\rho})\ + (\sin\phi\,\frac{d}{d\rho} + im\,\frac{\cos\phi}{\rho}) \right] \Phi = i\,\lambda_3\ e^{-i\phi}(\frac{d}{d\rho} + \frac{m}{\rho})\,\Phi\ ,$$

$$E_2+iB_2 = \lambda_3 \left[(+i\,(\sin\phi\,\frac{d}{d\rho} + im\,\frac{\cos\phi}{\rho}) - (\cos\phi\,\frac{d}{d\rho} - im\,\frac{\sin\phi}{\rho}) \right] \Phi = -\lambda_3\ e^{-i\phi}(\frac{d}{d\rho} + \frac{m}{\rho})\,\Phi\ .$$

It seems reasonable to expect further developments in this matrix based approach to Maxwell theory, as a possible base to explore general method to separate the variables for Maxwell equations in different coordinates. Also it would be desirable to extent this approach to Maxwell theory in curved space – time models.

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